Characterization of Almost (α, μ)-Continuous Functions and its properties

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Abstract

The purpose of this paper is to introduce the notion of almost (α, μ)-continuity in generalized topological spaces as a generalization of (λ, μ)-continuity and almost (α, μ)-continuity. Also we obtain several characterizations of almost (α, μ)-continuous functions and show that almost (α, μ)-continuity is equivalent to almost feeble (λ, μ)-continuity. In the last section, we obtain several properties of almost (α, μ)-continuous functions and a characterization of λ-α-irresolute functions.
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1. Introduction

In [2], [3], [4], A. Cs□sz□r founded the theory of generalized topological spaces, and studied the extremely elementary character of these classes. Especially he introduced the notions of continuous functions on generalized topological spaces, and investigated characterizations of generalized continuous functions (= (λ, μ)-continuous functions in [4]). In [5], [6], [8], W.K. Min introduced the notions of weak (λ, μ)-continuity, almost (λ, μ)-continuity, (α, μ)-continuous, (σ, μ)-continuous, (π, μ)-continuous and (β, μ)-continuous on generalized topological spaces. In [1], S.-Z. Bai and Y.-P. Zuo introduced the notion of λ-α-irresolute functions and investigated their properties and relationships between (λ, μ)-continuity, (resp. weak (λ, μ)-continuity, almost (λ, μ)-continuity, (α, μ)-continuity, (σ, μ)-continuity, (π, μ)-continuity and (β, μ)-continuity) and λ-α-irresoluteness.

The purpose of this paper is to introduce the notion of almost (α, μ)-continuity in generalized topological spaces as a generalization of (λ, μ)-continuity and almost (α, μ)-continuity. Also we obtain several characterizations of almost (α, μ)-continuous functions and show that almost (α, μ)-continuity is equivalent to almost feeble (λ, μ)-continuity. In the last section, we obtain several properties of almost (α, μ)-continuous functions and a characterization of λ-α-irresolute functions.

2. Preliminaries

Definition 2.1 [4]
Let X be a nonempty set and λ be a collection of subsets of X. Then λ is called a generalized topology (briefly GT) on X if $\phi \in \lambda$ and $G_i \in \lambda$ for $i \in I \neq \phi$ implies $G = \bigcup_{i \in I} G_i \in \lambda$. We say λ is strong if $X \in \lambda$, and we call the pair (X, λ) a generalized topological space (briefly GTS) on X.

The elements of λ are called λ-open sets and their complements are called λ-closed sets. The generalized closure of a subset S of X, denoted by $c_\lambda(S)$, is the intersection of λ-closed sets including S. And the interior of S, denoted by $i_\lambda(S)$, is the union of λ-open sets contained in S.
Definition 2.2 [3]

Let (X, λ) be a generalized topological space and $A \subset X$. Then A is said to be

(i) λ-semi-open if $A \subset c_\lambda(i_\lambda(A))$,

(ii) λ-preopen if $A \subset i_\lambda(c_\lambda(A))$,

(iii) λ-α-open if $A \subset i_\lambda(c_\lambda(i_\lambda(A)))$,

(iv) λ-β-open if $A \subset c_\lambda(i_\lambda(c_\lambda(A)))$,

(v) λ-regular open [8] if $A = i_\lambda(c_\lambda(A))$.

The complement of λ-semi-open (resp., λ-preopen, λ-α-open, λ-β-open, λ-regular open) is said to be λ-semi-closed (resp., λ-preclosed, λ-α-closed, λ-β-closed, λ-regular closed).

Let us denote by $\sigma(\lambda_X)$ (briefly σ_X or σ) the class of all λ-semi-open sets on X, by $\pi(\lambda_X)$ (briefly π_X or π) the class of all λ-preopen sets on X, by $\alpha(\lambda_X)$ (briefly α_X or α) the class of all λ-α-open sets on X, by $\beta(\lambda_X)$ (briefly β_X or β) the class of all λ-β-open sets on X, by $\rho(\lambda_X)$ (briefly ρ_X or ρ) the class of all λ-regular open sets on X.

Obviously [8] $\lambda_X \subset \alpha(\lambda_X) \subset \sigma(\lambda_X) \subset \beta(\lambda_X)$ and $\lambda_X \subset \alpha(\lambda_X) \subset \pi(\lambda_X) \subset \beta(\lambda_X)$.

Lemma 2.3 [3]

Let (X, λ) be a generalized topological space and $A \subset X$. Then A is λ-α-open if and only if it is λ-semi-open and λ-preopen.

Definition 2.4

Let (X, λ) and (Y, μ) be GTS's. Then a function $f : (X, \lambda) \rightarrow (Y, \mu)$ is said to be

(i) (λ, μ)-continuous [4] if for each μ-open set U in Y, $f^{-1}(U)$ is λ-open in X,

(ii) (α, μ)-continuous [6] if for each μ-open set U in Y, $f^{-1}(U)$ is λ-α-open in X,

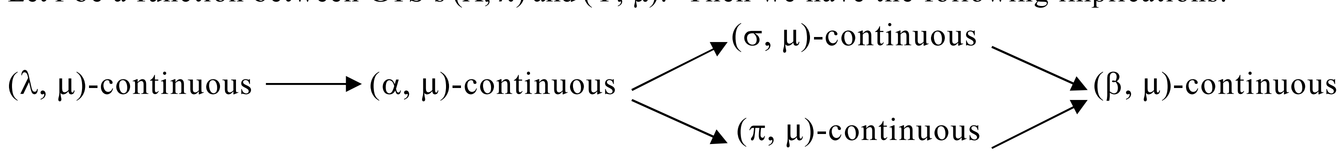
(iii) (σ, μ)-continuous [6] if for each μ-open set U in Y, $f^{-1}(U)$ is λ-semi-open in X,

(iv) (π, μ)-continuous [6] if for each μ-open set U in Y, $f^{-1}(U)$ is λ-preopen in X,

(v) (β, μ)-continuous [6] if for each μ-open set U in Y, $f^{-1}(U)$ is λ-β-open in X.

Remark 2.5 [6]

Let f be a function between GTS's (X, λ) and (Y, μ). Then we have the following implications.



Definition 2.6 [1]

Let (X, λ) and (Y, μ) be GTS's. Then a function $f : (X, \lambda) \rightarrow (Y, \mu)$ is said to be λ-α-irresolute if the inverse image of every μ-α-open set in Y is an λ-α-open set in X.

Remark 2.7 [3]

Let (X, λ) be a generalized topological space and $A \subset X$. The λ-α-closure (resp. λ-semi-closure, λ-preclosure, λ-β-closure) of a subset A of X, denoted by $c_\alpha(A)$ (resp. $c_\sigma(A)$, $c_\pi(A)$, $c_\beta(A)$), is the intersection of all λ-α-closed (resp. λ-semi-closed, λ-preclosed, λ-β-closed) sets containing A.

The λ-α-interior (resp. λ-semi-interior, λ-preinterior, λ-β-interior) of a subset A of X, denoted by $i_\alpha(A)$ (resp. $i_\sigma(A)$, $i_\pi(A)$, $i_\beta(A)$), is the union of all λ-α-open (resp. λ-semi-open, λ-preopen, λ-β-open) sets contained in A.

Definition 2.8 [7]

Let (X, λ) and (Y, μ) be GTS's. Then a function $f : (X, \lambda) \to (Y, \mu)$ is said to be (λ, μ)-open if the image of each λ-open set in X is an μ-open set of Y.

Definition 2.9 [5]

Let (X, λ) and (Y, μ) be GTS's. Then a function $f : (X, \lambda) \to (Y, \mu)$ is said to be almost (λ, μ)-continuous at $x \in \lambda$ if for each μ-open set V containing f(x), there exists a λ-open set U containing x such that $f(U) \subset i_\mu(c_\mu(V))$.

Remark 2.10 [5]

From the above definition of almost (λ, μ)-continuity, we have the following implications but the reverse relations may not be true in general.

(λ, μ)-continuous $\Rightarrow$ almost (λ, μ)-continuous

Definition 2.11 [8]

Let (X, λ) and (Y, μ) be GTS's. Then a function $f : (X, \lambda) \to (Y, \mu)$ is said to be weakly (λ, μ)-continuous if for each $x \in X$ and each μ-open set V containing f(x), there exists a λ-open set U containing x such that $f(U) \subset c_\mu(V)$.

3. Almost (α, μ)-Continuous Functions

Definition 3.1

Let (X, λ) and (Y, μ) be GTS's. Then a function $f : (X, \lambda) \to (Y, \mu)$ is said to be almost (α, μ)-continuous if for each μ-regular open set U in Y, $f^{-1}(U)$ is λ-α-open in X.

Remark 3.2

It is obvious that (α, μ)-continuity and almost (λ, μ)-continuity are independent of each other.

Remark 3.3

It is obvious that almost (α, μ)-continuity is implied by (α, μ)-continuity and almost (λ, μ)-continuity. However, by Remark 3.2 the converses are not true in general.

Definition 3.4

Let (X, λ) and (Y, μ) be GTS's. Then a function $f : (X, \lambda) \to (Y, \mu)$ is said to be (η, μ)-continuous if for every μ-regular open sets U, V of Y,

(i) $f^{-1}(V) \subset i_\lambda(c_\lambda(f^{-1}(V)))$,

(ii) $i_\lambda(c_\lambda(f^{-1}(U \cap V))) = i_\lambda(c_\lambda(f^{-1}(U))) \cap i_\lambda(c_\lambda(f^{-1}(V)))$

Remark 3.5

It is obvious that both (α, μ)-continuity and almost (λ, μ)-continuity imply (η, μ)-continuity.

Lemma 3.6

Let A and B be subsets of (X, λ). If either $A \in \sigma(\lambda_X)$ or $B \in \sigma(\lambda_X)$, then $i_\lambda(c_\lambda(A \cap B)) = i_\lambda(c_\lambda(A)) \cap i_\lambda(c_\lambda(B))$.

Theorem 3.7

Let (X, λ) and (Y, μ) be GTS's. If a function f : (X, λ) → (Y, μ) is almost (α, μ)-continuous, then it is (η, μ)-continuous.

Proof

Let U, V be any μ-regular open sets of (Y, μ). Since f is almost (α, μ)-continuous, $f^{-1}(V) \in \alpha(\lambda_X) \subset \pi(\lambda_X)$ and hence $f^{-1}(V) \subset i_\lambda(c_\lambda(f^{-1}(V)))$. Moreover, since $f^{-1}(V) \in \alpha(\lambda_X) \subset \sigma(\lambda_X)$, by Lemma 3.6 we obtain (ii) of Definition 3.4.

Remark 3.8

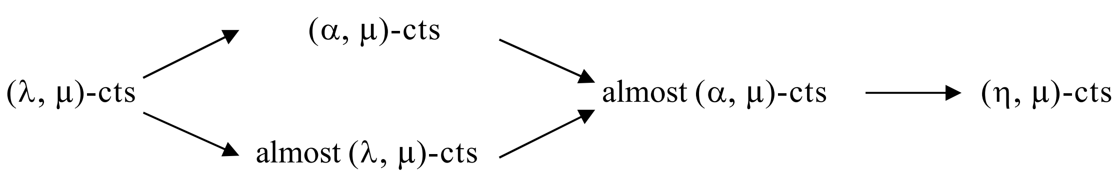
The converse of Theorem 3.7 is not true in general as the following example shows.

Example 3.9

Let X = {a, b, c, d}, λ = {ϕ, {d}, {a, c}, {a, c, d}}, Y = {p, q, r} and μ = {ϕ, {p}, {p, r}, {r}}. Define f : (X, λ) → (Y, μ) by f(a) = p; f(b) = f(c) = q and f(d) = r. Then f is (η, μ)-continuous but it is not almost (α, μ)-continuous, since {p} is μ-regular open in (Y, μ) and $\{a\} \notin \alpha(\lambda_X)$.

Remark 3.10

We have the following relationships, however, by Remarks 3.2 and 3.8, none of these implications is reversible.



4. Characterizations

Definition 4.1

Let (X, λ) and (Y, μ) be GTS's. A subset S is said to be feebly λ-open if there exists U ∈ X such that $U \subset S \subset c_\sigma(U)$. A function f : (X, λ) → (Y, μ) is said to be almost feebly (λ, μ)-continuous (resp., feebly (λ, μ)-continuous) if $f^{-1}(V)$ is feebly λ-open in X for every μ-regular open (resp., μ-open) set V of Y.

Lemma 4.2

Let U be a subset of a GTS (X, λ). Then $U \in \pi(\lambda_X)$ if and only if $i_\lambda(c_\lambda(U)) = c_\sigma(U)$.

Proof

Suppose that $U \in \pi(\lambda_X)$. It is clear that $i_\lambda(c_\lambda(U)) \subset c_\sigma(U)$ for every subset U of X. Let $x \notin i_\lambda(c_\lambda(U))$. Then $x \in X - i_\lambda(c_\lambda(U)) \in \sigma(\lambda_X)$ and $U \cap X - i_\lambda(c_\lambda(U)) = \phi$ since $U \in \pi(\lambda_X)$. This shows that $x \notin c_\sigma(U)$. Therefore, we obtain $c_\sigma(U) = i_\lambda(c_\lambda(U))$. The converse is obvious since $U \subset c_\sigma(U)$ for every subset U of X.

Lemma 4.3

Let U be a subset of a GTS (X, λ). Then $U \in \alpha(\lambda_X)$ if and only if U is feebly λ-open in X.

Proof

It is obvious that $U \in \alpha(\lambda_X)$ if and only if there exists G ∈ λ such that $G \subset U \subset i_\lambda(c_\lambda(G))$. Therefore, Lemma 4.2 completes the proof.

Theorem 4.4

Let (X, λ) and (Y, μ) be GTS's. If the function $f : (X, \lambda) \to (Y, \mu)$ is almost (α, μ)-continuous (resp., (α, μ)-continuous) if and only if it is almost feebly (λ, μ)-continuous (resp., feebly (λ, μ)-continuous).

Proof

This is an immediate consequence of Lemma 4.3.

Theorem 4.5

Let (X, λ) and (Y, μ) be GTS's. For a function $f : (X, \lambda) \to (Y, \mu)$, the following are equivalent:

(i) f is almost (α, μ)-continuous.

(ii) For each $x \in X$ and each $V \in \sigma$ containing f(x), there exists $U \in \alpha(\lambda_X)$ containing x such that $f(U) \subset i_\mu(c_\mu(V))$.

(iii) $f^{-1}(F)$ is λ-α-closed in (X, λ) for every μ-regular closed set F of (Y, μ).

Proof

This is obvious from the definitions.

Definition 4.6

The generalized topology having the λ-regular open sets in (X, λ) as a basis is called the λ-semi-regularization of (X, λ) and it is denoted by $s(\lambda_X)$.

Theorem 4.7

Let (X, λ) and (Y, μ) be GTS's. For a function $f : (X, \lambda) \to (Y, \mu)$, the following are equivalent:

(i) $f : (X, \lambda) \to (Y, \mu)$ is almost (α, μ)-continuous.

(ii) $f : (X, \lambda) \to (Y, s(\mu_X))$ is (α, μ)-continuous.

(iii) $f : (X, \alpha(\lambda_X)) \to (Y, \mu)$ is almost (λ, μ)-continuous.

(iv) $f : (X, \alpha(\lambda_X)) \to (Y, s(\mu_X))$ is (λ, μ)-continuous.

Proof

Every $V \in s(\mu_X)$ is the union of μ-regular open sets of (Y, μ). Therefore, (i) is equivalent to (ii). It is obvious that (i), (iii) and (iv) are all equivalent.

5. Some Properties

Lemma 5.1

Let A be a subset of a generalized topological space (X, λ). If either $A \in \sigma(\lambda_X)$ or $A \in \pi(\lambda_X)$, then $A \cap V$ is λ-α-open in the subspace $(A, \lambda/A)$ for every $V \in \alpha(\lambda_X)$.

Remark 5.2

Let (X, λ) and (Y, μ) be GTS's, and $f : (X, \lambda) \to (Y, \mu)$ be a function. If f is almost feebly (λ, μ)-continuous and A is λ-open in X, then the restriction $f \backslash A : A \to Y$ is almost feebly (λ, μ)-continuous. As an improvement of this result, we have the following.

Theorem 5.3

Let (X, λ) and (Y, μ) be GTS's and the function $f : (X, \lambda) \to (Y, \mu)$ is almost (α, μ)-continuous function. If either $A \in \sigma(\lambda_X)$ or $A \in \pi(\lambda_X)$, then the restriction $f \backslash A : (A, \lambda/A) \to (Y, \mu)$ is almost (α, μ)-continuous.

Proof

Let V be a μ-regular open set of (Y, μ). Since f is almost (α, μ)-continuous, $f^{-1}(V) \in \alpha(\lambda_X)$ and by Lemma 5.1, $f^{-1}(V) \cap A = (f\backslash A)^{-1}(V) \in \alpha(\lambda/A_X)$. Therefore f\A is almost (α, μ)-continuous.

Remark 5.4

Let (X, λ) and (Y, μ) be GTS's, and the function f : (X, λ) → (Y, μ) is an almost (λ, μ)-continuous injection and Y is μ-Hausdorff, then X is λ-Hausdorff. The following theorem is a slight improvement of this result.

Theorem 5.5

Let (X, λ) and (Y, μ) be GTS's, and the function f : (X, λ) → (Y, μ) is an almost (α, μ)-continuous injection and (Y, μ) is μ-Hausdorff, then (X, λ) is λ-Hausdorff.

Proof

Since f is almost (α, μ)-continuous, by Theorem 4.7, f : $(X, \alpha(\lambda_X)) \to (Y, \mu)$ is almost (λ, μ)-continuous. Since (Y, μ) is μ-Hausdorff, by Remark 5.4, so is $(X, \alpha(\lambda_X))$. Then it follows that (X, λ) is λ-Hausdorff.

Theorem 5.6

Let (X, λ) and (Y, μ) be GTS's, and f, g : (X, λ) → (Y, μ) are almost (α, μ)-continuous injection and (Y, μ) is μ-Hausdorff, then (x∈ X \ f(x) = g(x)} is λ-α-closed in (X, λ).

Proof

Since f, g are almost (α, μ)-continuous, by Theorem 4.7, f, g : $(X, \alpha(\lambda_X)) \to (Y, s(\mu_X))$ are (λ, μ)-continuous. Since $(Y, s(\mu_X))$ is μ-Hausdorff, (x∈ X \ f(x) = g(x)} is λ-closed in $(X, \alpha(\lambda_X))$ and hence it is λ-α-closed in (X, λ).

Definition 5.7

Let (X, λ) and (Y, μ) be GTS's. Then a function f : (X, λ) → (Y, μ) is said to be semi-weakly (λ, μ)-continuous if for each x∈X and each μ-open set V containing f(x), there exists a λ-semi-open set U containing x such that $f(U) \subset c_\sigma(V)$.

Lemma 5.8

Let (X, λ) and (Y, μ) be GTS's. A function f : (X, λ) → (Y, μ) is semi-weakly (λ, μ)-continuous if and only if $f^{-1}(V) \in \sigma(\lambda_X)$ for every μ-regular open set V of Y.

Proof

Necessity. Let V be a μ-regular open set V of Y. For each $x \in f^{-1}(V)$, there exists $U_X \in \sigma(\lambda_X)$ containing x such that $f(U_X) \subset c_\sigma(V)$. By Lemma 4.2, we have $c_\sigma(V) = i_\mu(c_\mu(V)) = V$ and hence $x \in U_X \subset f^{-1}(V)$. Therefore, it follows that $f^{-1}(V) \in \sigma(\lambda_X)$.

Sufficiency. Let x∈X and f(x)∈V∈σ. Put $U = x \in f^{-1}(i_\mu(c_\mu(V)))$, then $x \in U \in \sigma(\lambda_X)$ and $f(U) \subset i_\mu(c_\mu(V)) = c_\sigma(V)$ by Lemma 4.2. This shows that f is semi-weakly (λ, μ)-continuous.

Remark 5.9

Semi-weakly (λ, μ)-continuous and (η, μ)-continuous are independent of each other. In Example 3.9, f is (η, μ)-continuous but it is not semi-weakly (λ, μ)-continuous since {x} is μ-regular open in (Y, μ) and $\{a\} \notin \sigma(\lambda_X)$. Moreover, a semi-weakly (λ, μ)-continuous function is not necessarily (η, μ)-continuous as the following Example 5.10 shows.

Example 5.10

Let X = {a, b, c}, λ = {ϕ, {a}, {b}, {a, b}} and μ = {ϕ, {a}, {b, c}}. Then the identity function f : (X, λ) $\rightarrow$ (X, μ) is semi-weakly (λ, μ)-continuous. However, f is not (η, μ)-continuous, since {b, c} is μ-regular open in (X, μ) and f^{-1}({b, c}) does not satisfy (i) of Definition 3.4.

Theorem 5.11

Let (X, λ) and (Y, μ) be GTS's. If the function f : (X, λ) $\rightarrow$ (Y, μ) is almost (α, μ)-continuous if and only if it is (η, μ)-continuous and semi-weakly (λ, μ)-continuous.

Proof

Necessity. Suppose that f is almost (α, μ)-continuous. By Theorem 3.7, f is (η, μ)-continuous. Since $\alpha(\lambda_X)\subset\sigma(\lambda_X)$, by Lemma 5.8, f is semi-weakly (λ, μ)-continuous.

Sufficiency. Let V be any μ-regular open set in Y. Since f is semi-weakly (λ, μ)-continuous, by Lemma 5.8, $f^{-1}(V) \in\sigma(\lambda_X)$. Moreover, f is ($\eta$, μ)-continuous, $f^{-1}(V) \in\pi(\lambda_X)$ and hence $f^{-1}(V) \in \alpha(\lambda_X)$. Thus f is almost ($\alpha$, μ)-continuous.

Remark 5.12

Let A be a subset of a generalized topological space (X, λ). Then A $\in\alpha(\lambda_X)$ if and only if there exists a α-regular open set O in (X, λ) and a nowhere dense set N such that A = O-N.

Theorem 5.13

Let (X, λ) and (Y, μ) be GTS's. A function f : (X, λ) $\rightarrow$ (Y, μ) is λ-α-irresolute if and only if it is almost (α, μ)-continuous and $f^{-1}(N)$ is λ-α-closed in (X, λ) for every nowhere dense set N in (Y, μ).

Proof

Necessity. Assume that f is λ-α-irresolute. It is obvious that f is almost (α, μ)-continuous. Let N be nowhere dense in (Y, μ). Then, $i_\mu(c_\mu(N)) = \phi$ and $Y - N \subset Y = Y - i_\mu(c_\mu(N)) = c_\mu(i_\mu(Y-N))$. Therefore, we obtain $Y - N \subset i_\mu(c_\mu(i_\mu(Y-N)))$. This shows that $Y - N\in\alpha(\lambda_Y)$. Therfore, $f^{-1}(Y-N)\in\alpha(\lambda_X)$ and $f^{-1}(N)$ is λ-α-closed in (X, λ).

Sufficiency. Let $V\in\alpha(\lambda_Y)$. By Remark 5.12, V = O – N, where O is μ-regular open in (Y, μ) and N is nowhere dense in (Y, μ). By the hypothesis, $f^{-1}(O)\in\alpha(\lambda_X)$ and $f^{-1}(N)$ is λ-α-closed in (X, λ) and hence we have $f^{-1}(V) \subset f^{-1}(O) \cap (X - f^{-1}(N)) \in \alpha(\lambda_X)$. This shows that f is λ-α-irresolute.

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