Integral representation of differentiable functions and embedding theorem in variable Sobolev spaces

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Abstract. In this paper, the authors give an integral representation of functions from an anisotropic variable Sobolev space. The given integral representation is similar to the integral representation of functions from the anisotropic Sobolev space with a constant exponent. Next, we prove the boundedness of the differentiation operator from anisotropic variable Sobolev space to variable Lebesgue spaces. In particular, we proved the boundedness of the average function in variable Lebesgue space under the global log-Hölder continuity condition on variable exponents.

Keywords: variable Lebesgue space, variable anisotropic Sobolev space, integral representation, convolution operator, embedding theorem.

Mathematics Subject Classification (2010): 26D10, 26D15.

1 Introduction

The embedding theory of spaces of differentiable functions of several variables developed as a new direction in mathematics in the 1930s in the works of Sobolev. This theory studies important connections and relationships of differential properties of functions in various metrics. In addition to its independent interest from the point of view of functions, it has numerous and

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effective applications in the theory of partial differential equations (see [20]). In many problems of mathematical physics and variational calculus it is not sufficient to deal with the classical solutions of differential equations. It is necessary to introduce the notion of weak derivatives and to work in the so-called Sobolev spaces. Suppose that $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ is a multi-index and let $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Sobolev studied isotropic spaces $W_p^{(\ell)}(G)$ of functions $f(x)$ defined on a domain $G \subset \mathbb{R}^n$ with the norm

$$\sum_{|\alpha|\leq\ell} \|D^\alpha\|_{L_p(G)},$$

where $p \geq 1$ and $\ell \in \mathbb{N}$. Sobolev obtained embedding theorems for domains of n-dimensional spaces, namely theorems on summability to the q-th power of weak derivatives $D^\beta f$ with respect to a domain or to manifolds of lower dimension belonging to it. In subsequent years, embedding theory was intensively developed in various directions by many mathematicians and received new interesting and important applications(see [4], [16] and [17]).

Variable exponent Lebesgue spaces were first studied by Orlicz in 1931 (see [18]). Since the 1990s, variable exponent Lebesgue spaces and variable exponent Sobolev spaces have been used in a variety of fields, the most important of which is the mathematical modeling of electrorheological fluids. In 1997, the variable exponent Lebesgue spaces was applied to the study of image processing. Namely, in image reconstruction, the variable exponent interpolation technique can be used to obtain a smoother image. For the theory and applications of variable exponent Lebesgue spaces and variable exponent Sobolev spaces, see [5], [7], [10], [12], [14], [15], [19] and the references therein. Embedding is always a classical topic in functional analysis, partial differential equations and other fields. Related to embedding theorems, we refer to [1]-[3], [9], [11]-[13], [22] and the references therein. Afterwards, some scholars did further research on the theory and applications of these kinds of spaces (see [6], [8] and the references therein). These results provide the necessary framework for the study of variational problems and elliptic equations with non-standard $p(x)$-growth conditions.

The remainder of the paper is structured as follows. Section 2 contained some preliminaries along with the standard ingredients used in the proofs. We give an integral representation result for functions defined on Sobolev spaces in Section 3. Our principal assertions, concerning the embedding theorem in variable Sobolev spaces are formulated and proved in Section 4. Namely, we prove the boundedness of the differentiation operator from anisotropic variable Sobolev space to variable Lebesgue spaces. In particular, we proved the boundedness of the Sobolev average function in variable Lebesgue space under the global log-Hölder continuity condition on variable exponents.

2 Preliminaries

Let $\mathbb{R}^n$ denote n-dimensional Euclidean space and let $x \in \mathbb{R}^n$. Suppose that $p(x)$ is a Lebesgue measurable function with values in $[1, \infty)$. We suppose that $1 \le \underline{p} \le p(x) \le \overline{p} \le \infty$, where $\underline{p} := \operatorname{ess\,inf}_{x>0} p(x)$ and $\overline{p} := \operatorname{ess\,sup}_{x>0} p(x)$. We denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all Lebesgue measurable functions $p : \mathbb{R}^n \mapsto [1, \infty)$. Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, we define the conjugate exponent function $p'(\cdot)$ by the formula $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$. Let $\mathbb{N}$ be the set of natural numbers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Suppose that $k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$ is a multi-index and let $|k| = k_1 + \ldots + k_n$. Next, we will use the conventions: $\mathbf{1} = (1, \ldots, 1)$, $k! = k_1! \cdots k_n!$, $x = (x_1, \ldots, x_n)$, $x^k = x_1^{k_1} \cdots x_n^{k_n}$ and $(x, k) = \sum_{i=1}^{n} x_i k_i$. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_i > 0$ $(i = 1, \ldots, n)$ and $\varepsilon > 0$. We set $\varepsilon^\lambda = \left(\varepsilon^{\lambda_1}, \ldots, \varepsilon^{\lambda_n}\right)$ and $\dfrac{x}{\varepsilon^\lambda} = \left(\dfrac{x_1}{\varepsilon^{\lambda_1}}, \ldots, \dfrac{x_n}{\varepsilon^{\lambda_n}}\right)$. Let $I_{\varepsilon^\lambda} = \left\{x : \ |x_i| \le \varepsilon^{\lambda_i},\, i = 1, \ldots, n\right\}$ be a cube with center at 0. By D_{x_i} and D_i we denote the partial derivative with respect to the variable x_i and with respect to the i-th variable, respectively. Let χ_G be a characteristic function of $G \subset \mathbb{R}^n$.

Definition 1. *We denote by $L_{p(\cdot)}(\mathbb{R}^n)$ the space of all Lebesgue measurable functions f on $\mathbb{R}^n$ such that for some $\lambda_0 > 0$*

$$\int\limits_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda_0}\right)^{p(x)} dx < \infty.$$

The norm in variable Lebesgue spaces $L_{p(\cdot)}(\mathbb{R}^n)$ is defined by the following functional (see [7], [10] and [14])

$$\|f\|_{L_{p(\cdot)}(\mathbb{R}^n)} = \|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int\limits_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}.$$

If $p(x) \equiv p = const$, then $L_{p(\cdot)}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ is the classical Lebesgue space with constant exponent. Also,

$$\|f\|_{L_{p(\cdot)}(\mathbb{R}^n)} = \left(\int\limits_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

It is well known that $L_{p(\cdot)}(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|\cdot\|_{p(\cdot)}$ (see [7]).

Definition 2. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $k = (k_1, \dots, k_n)$ be a multi-index. A function $f \in L^1_{loc}(\Omega)$ is weakly differentiable, if there exists a function $g_k \in L^1_{loc}(\Omega)$ such that for every $\varphi \in C_0^\infty(\Omega)$*

$$\int\limits_{\Omega} f(x)\varphi^{(k)}(x)\, dx = (-1)^{|k|} \int\limits_{\Omega} g(x)\varphi(x)\, dx.$$

A function g is called the k-th weak derivative of f and we write $g = f^{(k)}$.

The following Lemma holds.

Lemma 1. *Let $1 \le p(x) < \infty$ be a measurable functions and let $f \in L_{p(\cdot)}(\mathbb{R}^n)$. Suppose that $\{f_j\}$ is a sequence of functions in $L_{p(\cdot)}(\mathbb{R}^n)$ having weak derivatives $f_j^{(k)} \in L_{p(\cdot)}(\mathbb{R}^n)$, $j = 1, \dots$.*

If $\left\|f_i^{(k)} - f_j^{(k)}\right\|_{p(\cdot)} \to 0$ $(i, j \to \infty)$, then $f^{(k)} \in L_{p(\cdot)}(\mathbb{R}^n)$ and $\left\|f_i^{(k)} - f^{(k)}\right\|_{p(\cdot)} \to 0$ as $i \to \infty$.

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Proof. Since $\left\|f_i^{(k)} - f_j^{(k)}\right\|_{p(\cdot)} \to 0 \; (i,j \to \infty)$ there exists a function $h \in L_{p(\cdot)}(\mathbb{R}^n)$ such that $\left\|f_i^{(k)} - h\right\|_{p(\cdot)} \to 0$ as $i \to \infty$. Since f_i has a weak derivative $f_i^{(k)}$ on $\mathbb{R}^n$, for any function $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have

$$\int\limits_{\mathbb{R}^n} f_i^{(k)}(x)\,\varphi(x)\,dx = (-1)^{|k|} \int\limits_{\mathbb{R}^n} f_i(x)\,\varphi^{(k)}(x)\,dx. \tag{1}$$

We observe that integration in (1) is carried out only over some compact $K \subset \mathbb{R}^n$. So, passing to the limit as $i \to \infty$ in (1), one has

$$\int\limits_{\mathbb{R}^n} h(x)\,\varphi(x)\,dx = (-1)^{|k|} \int\limits_{\mathbb{R}^n} f(x)\,\varphi^{(k)}(x)\,dx.$$

Thus, we have that $h = f^{(k)}$.

This completes the proof.

Definition 3. *[15] Let $1 \le p(x) < \infty$ be a measurable functions and let $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n$. We denote by $W^{\ell}_{p(\cdot)}(\mathbb{R}^n)$ the anisotropic Sobolev space of all real-valued functions $f \in L_{p(\cdot)}(\mathbb{R}^n)$ having weak derivatives $D_i^{\ell_i} f \in L_{p(\cdot)}(\mathbb{R}^n)$, $i = 1, \ldots, n$. This space is equipped with the norm*

$$\|f\|_{W^{\ell}_{p(\cdot)}(\mathbb{R}^n)} = \|f\|_{p(\cdot)} + \sum_{i=1}^{n} \left\|D_i^{\ell_i} f\right\|_{p(\cdot)}.$$

Definition 4. *[4] Let $s = (s_1, \ldots, s_n)$ be a vector with positive components. Suppose that $0 < h \le \infty$, $\varepsilon > 0$ and $a_i \ne 0$, $i = 1, \ldots, n$. An s-horn of radius h and opening ε is a set defined as follows:*

$$V(s) = V(s,h) = \bigcup_{0<v<h} \left\{ x : \; \frac{x_i}{a_i} > 0, \;\; v < \left(\frac{x_i}{a_i}\right)^{s_i} < (1+\varepsilon)v \;\; (i = 1, \ldots, n) \right\}.$$

For two measurable functions f and g, we define the convolution by

$$(f \star g)(x) := \int\limits_{\mathbb{R}^n} f(x-y)\,g(y)\,dy = \int\limits_{\mathbb{R}^n} f(y)\,g(x-y)\,dy.$$

Definition 5. *[10] Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that a function $p : \Omega \mapsto [1, \infty)$ is locally log-Hölder continuous on Ω, if there exists $C_1 > 0$ such that*

$$|p(x) - p(y)| \leq \frac{C_1}{-log(|x-y|)} \quad for \ \ all \quad 0 < |x-y| \leq \frac{1}{2}.$$

We say that p satisfies the log-Hölder decay condition if there exist $p_\infty \geq 1$ and a constant $C_2 > 0$ such that

$$|p(x) - p_\infty| \leq \frac{C_2}{log(e+|x|)} \quad for \ \ all \quad x \in \Omega.$$

We say that p is globally log-Hölder continuous in Ω if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition. The constants C_1 and C_2 are called the local log-Hölder constant and the log-Hölder decay constant, respectively. The maximum $\max\{C_1, C_2\}$ is just called the log-Hölder constant of p.

Let us define the following class of variable exponents

$$\mathcal{P}^{\log}(\Omega) := \left\{ p \in \mathcal{P}(\Omega) : \ \frac{1}{p} \ \text{is globally log-Hölder continuous} \right\}.$$

Definition 6. *[21] A function $\psi \in L_1(\mathbb{R}^n)$ with $\psi \geq 0$ is called bell shaped, if it is radially decreasing and radially symmetric. The function $\Phi(x) := \operatorname*{ess\,sup}_{|y|\geq|x|} |f(y)|$ is called the least bell shaped majorant of f.*

Let us define the function

$$\psi_{\varepsilon^\lambda}(x) := \frac{1}{\varepsilon^{|\lambda|}} \psi\left(\frac{x}{\varepsilon^\lambda}\right).$$

We need the following lemma.

Lemma 2. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and let $\psi \in L_1(\mathbb{R}^n)$. Suppose that the least bell shaped majorant Ψ of ψ is integrable. Then the inequality*

$$\|f \star \psi_{\varepsilon^\lambda}\|_{p(\cdot)} \leq C \|\Psi\|_1 \|f\|_{p(\cdot)}$$

holds for all $f \in L_{p(\cdot)}(\mathbb{R}^n)$, *where* C *constant depend only on* p, λ *and* n. *Moreover,*

$$|f \star \psi_{\varepsilon^\lambda}| \leq 2\,\|\Psi\|_1\,\|f\|_{p(\cdot)} \quad \textit{for} \quad \textit{all}\ f \in L^1_{loc}(\mathbb{R}^n).$$

We observe that Lemma 1 in the case when $\lambda = (1, \ldots, 1)$ is proved [10]. The proof is similar to the proof of Lemma 4.6.3 in [10].

We need the following Theorem.

Theorem 1. *[10] Let* $p, q, r \in \mathcal{P}(\mathbb{R}^n)$ *with* $p(x) \leq q(x) \leq r(x)$ *almost everywhere* $x \in \mathbb{R}^n$. *Then*

$$L_{p(\cdot)}(\mathbb{R}^n) \bigcap L_{r(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{q(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{p(\cdot)}(\mathbb{R}^n) + L_{r(\cdot)}(\mathbb{R}^n).$$

The embedding constants are at most 2. *More precisely, for* $g \in L_{q(\cdot)}(\mathbb{R}^n)$ *the functions* $g_0 := sgn\ g \max\{|g| - 1,\, 0\}$ *and* $g_1 := sgn\ g \min\{|g|,\, 1\}$ *satisfy* $g = g_0 + g_1$, $|g_0|, |g_0| \leq |g|$, $\|g_0\|_{p(\cdot)} \leq 1$ *and* $\|g_1\|_{r(\cdot)} \leq 1$.

Using Lemma 1 we can in fact get better control of $f \star \psi_{\varepsilon^\lambda}$ when ε is small as we show in the following theorem.

Theorem 2. *Let* $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ *and let* $\psi \in L_1(\mathbb{R}^n)$. *Suppose that the least bell shaped majorant* Ψ *of* ψ *is integrable and* $\int\limits_{\mathbb{R}^n} \psi(x)\,dx = 1$. *Then* $f \star \psi_{\varepsilon^\lambda} \to f$ *a.e. as* $\varepsilon \to 0$ *for* $f \in L_{p(\cdot)}(\mathbb{R}^n)$. *In addition, if* $\overline{p} < \infty$, *then*

$$\|f \star \psi_{\varepsilon^\lambda} - f\|_{p(\cdot)} \to 0 \quad \textit{for} \quad \varepsilon \to 0.$$

Proof. Let $f \in L_{p(\cdot)}(\mathbb{R}^n)$ with $\|f\|_{p(\cdot)} \leq 1$. By Theorem 1 we can split f into $f = f_0 + f_1$ with $f_0 \in L_1(\mathbb{R}^n)$ and $f_1 \in L_\infty(\mathbb{R}^n)$. From [21] we deduce that $f_j \star \psi_{\varepsilon^\lambda}(x) \to f_j(x)$ almost everywhere $x \in \mathbb{R}^n$, $j = 0, 1$. This proves $f \star \psi_{\varepsilon^\lambda}(x) \to f(x)$ almost everywhere $x \in \mathbb{R}^n$.

Let $\overline{p} < \infty$. It remains to prove that $\|f \star \psi_{\varepsilon^\lambda} - f\|_{p(\cdot)} \to 0$ for $\varepsilon \to 0$. Let $\delta > 0$ be arbitrary. Then by density of simple functions in $L_{p(\cdot)}(\mathbb{R}^n)$ (see [14]) we can find a simple function g with $\|f - g\|_{p(\cdot)} < \delta$. This implies that

$$\|f \star \psi_{\varepsilon^\lambda} - f\|_{p(\cdot)} \leq \|g \star \psi_{\varepsilon^\lambda} - g\|_{p(\cdot)} + \|(f-g) \star \psi_{\varepsilon^\lambda} - (f-g)\|_{p(\cdot)} = I_1 + I_2.$$

Since g is a simple function, we have that $g \in L_1(\mathbb{R}^n) \bigcap L_{\overline{p}}(\mathbb{R}^n)$. Thus the classical theorem on mollification (see [21]) implies that $\|g \star \psi_{\varepsilon^\lambda} - g\|_{L_1 \bigcap L_{\overline{p}}} \to 0$ for $\varepsilon \to 0$. Thus, by Theorem 1 $\|g \star \psi_{\varepsilon^\lambda} - g\|_{p(\cdot)} \to 0$ for $\varepsilon \to 0$. This proves that $I_1 \to 0$ for $\varepsilon \to 0$. On the other hand, Lemma 2 implies that

$$I_2 = \|(f-g) \star \psi_{\varepsilon^\lambda} - (f-g)\|_{p(\cdot)} \leq C \; \|f - g\|_{p(\cdot)} \leq C\,\delta.$$

This implies

$$\lim_{\varepsilon \to 0} sup \, \|f \star \psi_{\varepsilon^\lambda} - f\|_{p(\cdot)} \leq C\,\delta.$$

Since $\delta > 0$ is arbitrary, this yields $\lim\limits_{\varepsilon \to 0} sup \, \|f \star \psi_{\varepsilon^\lambda} - f\|_{p(\cdot)} \to 0$ as $\varepsilon \to 0$.

This complete the proof of Theorem 2.

We need the following Theorem.

Theorem 3. *[7] Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and let $\overline{p} < \infty$. If the sequence $\{f_m\}$ is such that $f_m \to f$ pointwise a.e., and there exists $g \in L_{p(\cdot)}(\mathbb{R}^n)$ such that $|f_m(x)| \leq g(x)$ a.e., then $f \in L_{p(\cdot)}(\mathbb{R}^n)$ and $\|f_m - f\|_{p(\cdot)} \to 0$ as $m \to \infty$.*

Corollary 1. *Let $f \in L_{p(\cdot)}(\mathbb{R}^n)$ and let G_m be a sequence of bounded measurable sets such that $G_m \subset G_{m+1} \subset \mathbb{R}^n$ and $\mathbb{R}^n = \bigcup\limits_{m=1}^{\infty} G_m$. Then*

$$\lim_{m \to \infty} \left\|f \chi_{G_m} - f\right\|_{p(\cdot)} = 0$$

Suppose that K is an infinitely differentiable function on $\mathbb{R}^n$ with compact support and let $\int\limits_{\mathbb{R}^n} K(x)\,dx = 1$. For the sake of simplicity, we will henceforth

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assume that $\operatorname{supp} K = S(K) \subset I_1$. Let us define the average function for the function f with the kernel K and the averaging parameter ε^λ by the formula

$$f_{\varepsilon^\lambda}(x) = \varepsilon^{-|\lambda|} \int\limits_{\mathbb{R}^n} f(x+y)\, K\left(\frac{y}{v^\lambda}\right)\, dy = \varepsilon^{-|\lambda|} \int\limits_{\mathbb{R}^n} f(y)\, K\left(\frac{y-x}{\varepsilon^\lambda}\right)\, dy.$$

We observe that $f_{\varepsilon^\lambda} \in C_0^\infty(\mathbb{R}^n)$ and

$$D_x^\alpha f_{\varepsilon^\lambda}(x) = (-1)^{|\alpha|}\, \varepsilon^{-|\lambda|-(\alpha,\lambda)|} \int\limits_{R^n} f(y)\, D^\alpha K\left(\frac{y-x}{\varepsilon^\lambda}\right)\, dy.$$

It is obvious that $f_{\varepsilon^\lambda}(x) = (f \star K_{\varepsilon^\lambda})(x)$. Thus, we have similar results for the average function f_{ε^λ}.

Corollary 2. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and let $\overline{p} < \infty$. Then $C_0^\infty(\mathbb{R}^n)$ is dense in $L_{p(\cdot)}(\mathbb{R}^n)$.*

Proof. Suppose $f \in L_{p(\cdot)}(\mathbb{R}^n)$ and fix $\varepsilon > 0$. By Corollary 1, for functions from spaces $L_{p(\cdot)}(\mathbb{R}^n)$ there exists a bounded open set $G \subset \overline{G} \subset \mathbb{R}^n$ such that

$$\|f - f\chi_G\|_{p(\cdot)} < \frac{\varepsilon}{2}.$$

We set $f_G = f\chi_G$. For the function f_G, we compose the average function $f_{G,v^1} = f_{G,v}$, where $v > 0$. It is obvious that $f_{G,v} \in C_0^\infty(\mathbb{R}^n)$. By Theorem 2 for sufficiently small v we have

$$\|f_G - f_{G,v}\|_{p(\cdot)} < \frac{\varepsilon}{2}.$$

Thus, we get

$$\|f - f_{G,v}\|_{p(\cdot)} \le \|f - f_G\|_{p(\cdot)} + \|f_G - f_{G,v}\|_{p(\cdot)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This complete the proof.

3 The integral representation of functions in $W^{\ell}_{p(\cdot)}(\mathbb{R}^n)$.

The proof of the integral representation of functions given in this section is based on the following simple idea.

For a given locally summable function f, we construct its average function $f_{\varepsilon^\lambda}(x)$ with some kernel $\mathcal{L}$ and averaging parameter ε^λ, where λ is a fixed vector. It is obvious that the average function $f_{\varepsilon^\lambda}(x)$ can be considered as a continuously differentiable function with respect to the parameter ε.

We set $0 < \varepsilon < h$. By the fundamental theorem of calculus, we have

$$f_{\varepsilon^\lambda}(x) = f_{h^\lambda}(x) - \int_\varepsilon^h \frac{\partial}{\partial v} f_{v^\lambda}(x)\, dv.$$

The last identity is the initial one when deriving the integral representation of functions from the space $W^{\ell}_{p(\cdot)}(\mathbb{R}^n)$.

Let θ be the Heaviside function on $\mathbb{R}$. Proceeding in the same way as in the proof of the integral representation of functions in the case of constant exponents (see [4]), we can show that

$$f_{\varepsilon^\lambda}(x) = f_{h^\lambda}(x) + \int_\varepsilon^h \sum_{i=1}^n v^{-1-|\lambda|+\ell_i\lambda_i}\, dv \int_{\mathbb{R}^n} D_i^{\ell_i} f(x+y)\, \mathcal{L}_i\left(\frac{y}{v^\lambda}\right)\, dy, \qquad (2)$$

where $f_{\varepsilon^\lambda}(x) = \varepsilon^{-|\lambda|} \int_{\mathbb{R}^n} f(x+y)\, \Omega\left(\frac{y}{v^\lambda}\right)\, dy$,

$$\Omega(x) = D_x^k \left[\frac{x^{k-\mathbf{1}}}{(k-\mathbf{1})!} \int_{\mathbb{R}^n} K(z) \prod_{j=1}^n \theta\left(x_j - z_j\right)\, dz \right],$$

$k = (k_1, \dots, k_n)$, k_i are sufficiently large natural numbers, $\mathcal{L}_i(x) = (-1)^{\ell_i}\, \lambda_i\, D_i^{k_i-\ell_i} \widetilde{\mathcal{L}}_i(x)$,

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$\ell_i \leq k_i$ $(i = 1, \ldots, n)$ and

$$\widetilde{\mathcal{L}}_i(x) = D^{k-k_i e_i} \left[\frac{x^{\mathbf{k-1}}\, x_i}{(k-\mathbf{1})!} \int\limits_{\mathbb{R}^{n-1}} K\left(z_1, \ldots, x_i, \ldots, z_n\right) \left(\prod_{\substack{j \\ j\neq i}} \theta\left(x_j - z_j\right) \right) dz^{(i)} \right].$$

Representation (2) can be considered as an integral representation of the difference in the values of the average functions with parameters ε^λ and h^λ at the point x through the integrals of the weak derivatives of the functions f.

So, we have the following theorem.

Theorem 4. *Suppose that $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{N}_0^n$ and let $h > 0$. Let $p \in \mathcal{P}^{\log}\left(\mathbb{R}^n\right)$ and let $K \in C_0^\infty\left(\mathbb{R}^n\right)$. Suppose that the least bell shaped majorant Φ of K is integrable and $\int\limits_{\mathbb{R}^n} K(x)\, dx = 1$.*

Then for any function $f \in W^\ell_{p(\cdot)}\left(\mathbb{R}^n\right)$ the following integral representation holds

$$f(x) = f_{h^\lambda}(x) + \int\limits_0^h \sum_{i=1}^n v^{-1-|\lambda|+\ell_i\lambda_i}\, dv \int\limits_{\mathbb{R}^n} D_i^{\ell_i} f(x+y)\, \mathcal{L}_i\left(\frac{y}{v^\lambda}\right) dy. \quad (3)$$

Proof. It is clear that all conditions of Theorem 2 are satisfied. Thus, by tending ε to 0 in (2), by Theorem 2 we obtain identity (3) for almost everywhere $x \in \mathbb{R}^n$. We observe that the support of the representation (3) is the shifted ℓ-horn $x + V(\ell)$.

This completes the proof.

4 Embedding theorems in variable exponent anisotropic Sobolev space

In this section we prove the continuity of embedding operator on variable exponent anisotropic Sobolev space.

Theorem 5. *Suppose that* $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ *with* $\overline{p} < \infty$ *and let* $h > 0$. *Let* $\alpha = (\alpha_1, \ldots, \alpha_n) \in N_0^n$ *be a multi-index,* $\left|\frac{\alpha}{\ell}\right| < 1$ *and* $\lambda = \frac{\mathbf{1}}{\ell}$. *Suppose that all the conditions of Theorem 2 and Theorem 4 are satisfied.*

Then $D^\alpha W^\ell_{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L_{p(\cdot)}(\mathbb{R}^n)$ *and the following inequality holds*

$$\|D^\alpha f\|_{p(\cdot)} \leq C_1 h^{-\left|\frac{\alpha}{\ell}\right|} \|f\|_{p(\cdot)} + C_2\, h^{1-\left|\frac{\alpha}{\ell}\right|} \sum_{i=1}^{n} \left\|D_i^{\ell_i} f\right\|_{p(\cdot)},$$

where C_1, C_2 *is independent on* f *and* $h \in (0, h_0)$.

Proof. It is obvious that for $\lambda = \frac{\mathbf{1}}{\ell}$ the representation (2) has the form

$$f_{\varepsilon^\lambda}(x) = f_{h^\lambda}(x) + \int_{\varepsilon}^{h} \sum_{i=1}^{n} v^{-|\lambda|}\, dv \int_{\mathbb{R}^n} D_i^{\ell_i} f(x+y)\, \mathcal{L}_i\left(\frac{y}{v^\lambda}\right) dy. \tag{4}$$

Applying the differentiation operator D_x^α to both parts of (4), we have

$$D_x^\alpha f_{\varepsilon^\lambda}(x) = D_x^\alpha f_{h^\lambda}(x) + \int_{\varepsilon}^{h} \sum_{i=1}^{n} v^{-|\lambda|-\left|\frac{\alpha}{\ell}\right|}\, dv \int_{\mathbb{R}^n} D_i^{\ell_i} f(x+y)\, M_i\left(\frac{y}{v^\lambda}\right) dy.$$

Next, we have

$$D_x^\alpha f_{\varepsilon^\lambda}(x) = (-1)^{|\alpha|} h^{-\left|\frac{\alpha}{\ell}\right|} \left(f \star (D^\alpha \Omega)_{h^\lambda}\right)(x) + \int_{\varepsilon}^{h} \sum_{i=1}^{n} v^{-\left|\frac{\alpha}{\ell}\right|} \left(D_i^{\ell_i} f \star (M_i)_{v^\lambda}\right)(x)\, dv. \tag{5}$$

We observe that $\Omega,\ M_i \in C_0^\infty(\mathbb{R}^n)$ $(i = 1, \ldots, n)$, and their supports are such that the support of the integral representation (4) is the ℓ-horn $V(\ell)$. In addition, let $\widetilde{\Omega}(x) = \operatorname*{ess\,sup}_{|y| \geq |x|} |D^\alpha \Omega(y)|$ and $N_i(x) = \operatorname*{ess\,sup}_{|y| \geq |x|} |M_i(y)|$ $(i = 1, \ldots, n)$, respectively. Let us assume that the functions $\widetilde{\Omega}$ and N_i satisfy the conditions of Theorem 2.

Let $0 < \varepsilon < \eta < h$. By generalized Minkowski inequality and by Lemma 2, we have

$$\left\|D_x^\alpha f_{\varepsilon^\lambda} - D_x^\alpha f_{\eta^\lambda}\right\|_{p(\cdot)} \leq \sum_{i=1}^{n} \int_{\varepsilon}^{\eta} v^{-\left|\frac{\alpha}{\ell}\right|} \left\|D_i^{\ell_i} f \star (M_i)_{v^\lambda}\right\|_{p(\cdot)} dv$$

$$\leq 2\sum_{i=1}^{n}\|N_i\|_1\left\|D_i^{\ell_i}f\right\|_{p(\cdot)}\int_{\varepsilon}^{\eta}v^{-\left|\frac{\alpha}{\ell}\right|}dv\leq C\left(\eta^{1-\left|\frac{\alpha}{\ell}\right|}-\varepsilon^{1-\left|\frac{\alpha}{\ell}\right|}\right)\sum_{i=1}^{n}\left\|D_i^{\ell_i}f\right\|_{p(\cdot)}$$

$$\leq C\,\eta^{1-\left|\frac{\alpha}{\ell}\right|}\sum_{i=1}^{n}\left\|D_i^{\ell_i}f\right\|_{p(\cdot)}. \qquad (6)$$

So, $D^{\alpha}f_{\varepsilon^{\lambda}}(x)$ is converges in $L_{p(\cdot)}(\mathbb{R}^n)$ as $\varepsilon\to 0$. On the other hand by Theorem 2 $\|f_{\varepsilon^{\lambda}}-f\|_{p(\cdot)}\to 0$ as $\varepsilon\to 0$. By Lemma 1 there are exists a weak derivative $D^{\alpha}f\in L_{p(\cdot)}(\mathbb{R}^n)$ and $\|D^{\alpha}f-D^{\alpha}f_{\varepsilon^{\lambda}}\|_{p(\cdot)}\to 0$ as $\varepsilon\to 0$. Thus, $\lim_{\varepsilon\to 0}\|D^{\alpha}f_{\varepsilon^{\lambda}}\|_{p(\cdot)}=\|D^{\alpha}f\|_{p(\cdot)}$.

By (5) and (6), we have

$$\|D^{\alpha}f\|_{p(\cdot)}\leq h^{-\left|\frac{\alpha}{\ell}\right|}\left\|(f\star(D^{\alpha}\Omega)_{h^{\lambda}})\right\|_{p(\cdot)}+C\,h^{1-\left|\frac{\alpha}{\ell}\right|}\sum_{i=1}^{n}\left\|D_i^{\ell_i}f\right\|_{p(\cdot)}.$$

By Lemma 2, we get

$$\|D^{\alpha}f\|_{p(\cdot)}\leq 2\left\|\widetilde{\Omega}\right\|_1 h^{-\left|\frac{\alpha}{\ell}\right|}\|f\|_{p(\cdot)}+C\,h^{1-\left|\frac{\alpha}{\ell}\right|}\sum_{i=1}^{n}\left\|D_i^{\ell_i}f\right\|_{p(\cdot)}. \qquad (7)$$

It is obvious that the inequality (7) holds for all $h\in(0,h_0)$.

We consider the function defined by

$$g(h)=h^{-\left|\frac{\alpha}{\ell}\right|}\|f\|_{p(\cdot)}+h^{1-\left|\frac{\alpha}{\ell}\right|}\sum_{i=1}^{n}\left\|D_i^{\ell_i}f\right\|_{p(\cdot)}.$$

The derivative of the function g is

$$g'(h)=h^{-1-\left|\frac{\alpha}{\ell}\right|}\left(-\left|\frac{\alpha}{\ell}\right|\|f\|_{p(\cdot)}+h\left(1-\left|\frac{\alpha}{\ell}\right|\right)\sum_{i=1}^{n}\left\|D_i^{\ell_i}f\right\|_{p(\cdot)}\right).$$

So, at $h=\dfrac{\left|\frac{\alpha}{\ell}\right|}{1-\left|\frac{\alpha}{\ell}\right|}\dfrac{\|f\|_{p(\cdot)}}{\sum_{i=1}^{n}\left\|D_i^{\ell_i}f\right\|_{p(\cdot)}}$ the function g has minimum value. Taking into account the value of h on the right side of inequality (7), we have

$$\|D^{\alpha}f\|_{p(\cdot)}\leq C\,\|f\|_{p(\cdot)}^{1-\left|\frac{\alpha}{\ell}\right|}\left(\sum_{i=1}^{n}\left\|D_i^{\ell_i}f\right\|_{p(\cdot)}\right)^{\left|\frac{\alpha}{\ell}\right|}. \qquad (8)$$

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Let $1 < s < \infty$ and let $a, b \geq 0$. Suppose that $s' = \dfrac{s}{s-1}$. Then the following Young's inequality holds

$$ab \leq \frac{a^s}{s} + \frac{b^{s'}}{s'}.$$

Let $\delta(\alpha, \ell) = \max\left\{1 - \left|\dfrac{\alpha}{\ell}\right|, \left|\dfrac{\alpha}{\ell}\right|\right\}$. Suppose that $s = \dfrac{1}{1 - \left|\frac{\alpha}{\ell}\right|}$. By (8) and by Young's inequality we have that

$$\|D^\alpha f\|_{p(\cdot)} \leq C\, \|f\|_{p(\cdot)}^{1-\left|\frac{\alpha}{\ell}\right|} \left(\sum_{i=1}^{n} \left\|D_i^{\ell_i} f\right\|_{p(\cdot)}\right)^{\left|\frac{\alpha}{\ell}\right|}$$

$$\leq C\, \left(\left(1 - \left|\frac{\alpha}{\ell}\right|\right) \|f\|_{p(\cdot)} + \left|\frac{\alpha}{\ell}\right| \sum_{i=1}^{n} \left\|D_i^{\ell_i} f\right\|_{p(\cdot)}\right)$$

$$\leq C\, \delta(\alpha, \ell) \left(\|f\|_{p(\cdot)} + \sum_{i=1}^{n} \left\|D_i^{\ell_i} f\right\|_{p(\cdot)}\right) = C\, \delta(\alpha, \ell)\, \|f\|_{W^{\ell}_{p(\cdot)}(\mathbb{R}^n)}.$$

This completes the proof.

Remark 1. *We observe that when $p(x) = p = const$, Theorem 5 was proved in [4]. Related to continuous and compact embeddings between different variable Sobolev and variable Lebesgue spaces, we refer to [1]-[3], [9], [11]-[13], [22] and the references therein.*

5 Competing interests.

The authors declare that they have no competing interests.

6 Funding Information.

There is no funding.

7 Authors contributions.

All authors contributed equally to this article. They read and approved the final version of the manuscript.

References

[1] R.A. Bandaliev, *The embedding theorem for anisotropic Sobolev spaces with variable exponent of summability*, Sci. Pedag. News Odlar Yurdu Univ., Ser. Phys., Math., Tech. Nat. Sci., 13(2)(2005), 138-142.

[2] R.A. Bandaliev, *On an inequality in Lebesgue space with mixed norm and with variable summability exponent*, Math. Notes, 84(3)(2008), 303-313.

[3] R.A. Bandaliev, *Embedding beetween variable exponent Lebesgue spaces with measures,* Azerb. J. Math., 2(1)(2012), 119-126.

[4] O.V. Besov, V.P. Ilin, S.M. Nikolskii, *Integral Representations of Functions and Imbedding Theorems,* vol. 1, John Wiley & Sons, 1978.

[5] Y. Chen, S. Levine, M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math., 66(2006), 1383-1406.

[6] Y. Cheng, B. Ge, R. P. Agarwal, *Variable-order fractional Sobolev spaces and nonlinear elliptic equations with variable exponents*, J. Math. Phys., 61(2020), 071507.

[7] D. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue spaces: Foundations and harmonic analysis*, Birkhäuser/ Springer, Heidelberg, 2013.

[8] L.M. Del Pezzo, J.D. Rossi, *Traces for fractional Sobolev spaces with variable exponents*, Adv. Oper. Theory, 2(2017), 435-446.

[9] L. Diening, *Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces* $L^{p(\cdot)}$ *and* $W^{k,p(\cdot)}$, Math. Nachr., 268(2004), 31-43.

[10] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, Springer-Verlag, Heidelberg, 2011.

[11] D. E. Edmunds, J. Rákosnik, *Sobolev embeddings with variable exponent*, Studia Math., 143(2000), 267293.

[12] X.L. Fan, D. Zhao, *On the spaces* $L^{p(x)}(\Omega)$ *and* $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl., 263(2001), 424-446.

[13] P. Hajlasz, P. Koskela, H. Tuominen, *Sobolev embeddings, extensions and measure density condition*, J. Funct. Anal., 254(2008), 1217-1234.

[14] V. Kokilashvili, A. Meskhi, H. Rafeiro, S. Samko, *Integral Operators in Non-Standard Function Spaces*, Vol.1. Variable exponent Lebesgue and Amalgam spaces. Oper. Theory Adv. Appl., 248, Birkhäuser, Heidelberg, 2016.

[15] O. Kováčik, J. Rákosnik, *On spaces* $L^{p(x)}$ *and* $W^{k,p(x)}$, Czechoslovak Math. J. 116(41) (1991), 592-618.

[16] V.G. Maz'ya, *Sobolev spaces,* Springer-Verlag, Berlin, 1985.

[17] A.M. Najafov, N.R. Rustamova, *On properties of functions from Sobolev-Morrey type spaces with dominant mixed derivatives*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics 37(4)(2017), 132-141.

[18] W. Orlicz, *Über konjugierte exponentenfolgen*, Studia Math.**3**(1931) 200-212.

[19] I.I. Sharapudinov, *On the topology of the space* $L^{p(t)}([0;1])$, Math. Notes, 26(3-4)(1976), 796-806.

[20] S.L. Sobolev, *Some applications of functional analysis in mathematical physics*, Translation Math. Monog., Vol.90, Amer. Math. Soc., 1991.

[21] E. Stein, *Singular integrals and differentiability properties of functions*, Princeton Math. Ser., No. 30. Princeton Univ. Press, Princeton, N.J., 1970.

[22] Y. Zhou, *Fractional Sobolev extension and imbedding*, Trans. Amer. Math. Soc., 367 (2015), 959979.