General integer solution of the equation

$a_1x_1 + a_2x_2 + a_3x_3 + \cdots . + a_nx_n = c$

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ABSTRACT. It is well known that the equation $ax + by = c$ where $a, b, c \in \mathbb{Z}$ has general integer solution if g,c,d of a and b i.e $g.c.d(a, b)$ divides c. Here we extend the result for n variables i.e we have find the intregal solution of the equation $a_1x_1 + a_2x_2 + a_3x_3 + \cdots . + a_nx_n = c$,where $c, a_i \epsilon \mathbb{Z} \forall 1 \leq i \leq n$.

Key Words: Integer, g.c.d , Bezout identity

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1.INTRODUCTION

By Bezout identity [1,2] we get the general integer solution of the equation $ax + by = c$ where $a, b, c \in \mathbb{Z}$ if g.c.d(a,b) divides c. Similarly $a_1x_1 + a_2x_2 + a_3x_3 + \cdots . + a_nx_n = c$, where $c, a_i \epsilon \mathbb{Z} \forall 1 \leq i \leq n$ has integer solution if $g.c.d(a_1, a_2, a_3, \ldots\ldots, a_n)$ divides c. It is the generalization of Bezout identity[1,2]. Here we first find the general integer solutions for n=2,3,4,5,6,7 etc. From these solution we try to find a general format for n variables. Then using induction we prove the required result.

2.PRELIMINARIES

When n increases from 2,3,… etc the solutions x_i where $1 \leq i \leq n$ have big structure compare to previous. To compact or reduce the sizes we introduce some notation which are given below. We write

(1) $g.c,d(a_1, a_2, a_3, \ldots\ldots, a_n) = g_n$ therefore $g.c.d(g_n, a_{n+1}) = g_{n+1}$. Also $(a_1, a_2, a_3, \ldots\ldots, a_n)$ means g.c.d of $(a_1, a_2, a_3, \ldots\ldots, a_n)$. e.g $(a_1, a_2) = g_2$; $(a_1, a_{2,}, a_3) = g_3$ etc.

(2) $\boldsymbol{L_n^m = l^m l^{m-1} \ldots . l^n}(m \geq n)$; $l^i \in Z \forall i$. e,g $L_2^5 = l^5 l^4 l^3 l^2$, $\boldsymbol{L_{r-2}^{n-3} = l^{n-3} l^{n-2} \ldots l^{r-2}}$, $L_m^m = l^m$, $l^{m+1} L_n^m = L_n^{m+1}$ etc.

3. RESULTS AND DISCUSSION

We 1st find the solution for $n = 2,3,4,5,6,7$. Here $c, a_i \in Z$ where $\forall 1 \leq i \leq n$. And $g.c,d(a_1, a_2, a_3, \ldots\ldots, a_n)|c$ or $g_n|c$. Now we find the solutions.

(1) $a_1x_1 + a_2x_2 = c$

The equation has integer solution if $g_2|c$. Again by Bezout's lemma[1] if g.c.d$(a_1, a_2) = g_2|c$ then there exist $x_1^0, x_2^0 \in Z$ such that $a_1x_1^0 + a_2x_2^0 = g_2$.

Now $a_1x_1 + a_2x_2 = c^/ g_2$ where $c = c^/ g_{2,} c^/ \in Z$. Therefore

$$a_1x_1 + a_2x_2 = c/g_2 \Rightarrow a_1x_1 + a_2x_2 = c/(a_1x_1^0 + a_2x_2^0) \Rightarrow a_1\big(x_1 - c/x_1^0\big)$$
$$= a_2(c/x_2^0 - x_2)$$
$$\Rightarrow x_1 - c/x_1^0 = k_1a_2;\ c/x_2^0 - x_2 = k_1a_1 \Rightarrow x_1 = c/x_1^0 + k_1a_2; x_2 = c/x_2^0 - k_1a_1$$

Hence

$$\Rightarrow x_1 = \frac{c}{g_2}x_1^0 + k_1a_2$$

$x_2 = \frac{c}{g_2}x_2^0 - k_1a_1$ where $k_1 \epsilon Z$ is arbitrary and $a_1x_1^0 + a_2x_2^0 = g_2$.

(2) $a_1x_1 + a_2x_2 + a_3x_3 = c$

Clearly as above it has integer solution if $g_3|c$. Now $a_1x_1 + a_2x_2 = c - a_3x_3$. So $g_2|c - a_3x_3$. Therefore $c - a_3x_3 = g_2l$ for some $l \in Z$. Now $a_1x_1 + a_2x_2 = (a_1x_1^0 + a_2x_2^0)l \Rightarrow$ $a_1(x_1 - lx_1^0) + a_2(x_2 - lx_2^0) = 0 \Rightarrow a_1X_1 + a_2X_2 = 0$ where $X_1 = x_1 - lx_1^0, X_2 = x_2 - lx_2^0$.

Now from (1.) as $g_2|0$ we have

$$X_1 = \frac{0}{g_2}x_1^0 + k_1a_2 = k_1a_2 \Rightarrow x_1 = lx_1^0 + k_1a_2\ ,$$
$$X_2 = \frac{0}{g_2}x_1^0 - k_1a_1 = -k_1a_1 \Rightarrow x_1 = lx_1^0 + k_1a_1$$

Again $g_2l + a_3x_3 = c$. As we want integer solution so $g.c.d(g_2, a_3) = g_3|c$. So again using (1) we have

$$l = \frac{c}{g_3}l^0 + k_2a_3\ , x_3 = \frac{c}{g_3}x_3^0 - k_2g_2$$

where $g_2l^0 + a_3x_3^0 = g_3$. Therefore

$$x_1 = \left(\frac{c}{g_3}l^0 + k_2a_3\right)x_1^0 + k_1a_2 = \left(\frac{c}{g_3}L_0^0 + k_2a_3\right)x_1^0 + k_1a_2$$

$$x_2 = \left(\frac{c}{g_3}l^0 + k_2a_3\right)x_2^0 - k_1a_1 = \left(\frac{c}{g_3}L_0^0 + k_2a_3\right)x_2^0 - k_1a_1$$

$$x_3 = \frac{c}{g_3}x_3^0 - k_2g_2$$

Where $k_i \in Z, 1 \le i \le 2$ arbitrary and

$$a_1x_1^0 + a_2x_2^0 = g_2$$

$$g_2l^0 + a_3x_3^0 = g_3 \Rightarrow a_1(x_1^0l^0) + a_2(x_2^0l^0) + a_3x_3^0 = g_3 \Rightarrow a_1(L_0^0x_1^0) + a_2(L_0^0x_2^0) + a_3x_3^0 = g_3$$

(3) $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = c$

Clearly $g_4|c$. Now $a_1x_1 + a_2x_2 + a_3x_3 = c - a_4x_4$. So $g_3|c - a_4x_4$. Therefore $c - a_4x_4 = g_3l$ for some $l \in Z$. Now

$$a_1x_1 + a_2x_2 + a_3x_3 = c - a_4x_4 = g_3l = \{a_1(L_0^0x_1^0) + a_2(L_0^0x_2^0) + a_3x_3^0\}l$$

$$a_1(x_1 - lL_0^0x_1^0) + a_2(x_2 - lL_0^0x_2^0) + a_3(x_3 - lx_3^0) = 0 \Rightarrow a_1X_1 + a_2X_2 + a_3X_3 = 0$$

where $X_1 = x_1 - lL_0^0x_1^0;\ X_2 = x_2 - lL_0^0x_2^0;\ X_3 = x_3 - lx_3^0$
So by using (3) as $g_3|0$ we have

$$X_1 = k_2a_3x_1^0 + k_1a_2 \Rightarrow x_1 = (lL_0^0 + k_2a_3)x_1^0 + k_1a_2$$
$$X_2 = k_2a_3x_2^0 - k_1a_1 \Rightarrow x_2 = (lL_0^0 + k_2a_3)x_2^0 - k_1a_1$$
$$X_3 = k_2g_2 \Rightarrow x_3 = lx_3^0 - k_2g_2$$

Again $g_3l + a_4x_4 = c$. As we want integer solution so $g.c.d(g_3, a_4) = g_4|c$. So again using (1) we have $l = \frac{c}{g_4}l^1 + k_3a_4;\ x_4 = \frac{c}{g_4}x_4^0 - k_3g_3$ where $g_3l^1 + a_4x_4^0 = g_4$.
Therefore

$$x_1 = \left(\frac{c}{g_4}l^1L_0^0 + l^0k_3a_4 + k_2a_3\right)x_1^0 + k_1a_2 = \left(\frac{c}{g_4}L_0^1 + L_0^0k_3a_4 + k_2a_3\right)x_1^0 + k_1a_2$$
$$x_2 = \left(\frac{c}{g_4}l^1L_0^0 + l^0k_3a_4 + k_2a_3\right)x_2^0 - k_1a_1 = \left(\frac{c}{g_4}L_0^1 + L_0^0k_3a_4 + k_2a_3\right)x_2^0 - k_1a_1$$
$$x_3 = \left(\frac{c}{g_4}l^1 + k_3a_4\right)x_3^0 - k_2g_2 = \left(\frac{c}{g_4}L_1^1 + k_3a_4\right)x_3^0 - k_2g_2$$

$$x_4 = \frac{c}{g_4}x_4^0 - k_3g_3$$

Where $k_i \in Z, 1 \le i \le 3$ arbitrary and

$$g_3l^1 + a_4x_4^0 = g_4$$

$$\Rightarrow a_1(L_0^1x_1^0) + a_2(L_0^1x_2^0) + a_3(L_1^1x_3^0) + a_4x_4^0 = g_4.$$

(4) $\boldsymbol{a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 = c}$

Using the same technique as above we get the general integer solution where

$$x_1 = \left(\frac{c}{g_4}L_0^2 + L_0^1k_4a_5 + L_0^0k_3a_4 + k_2a_3\right)x_1^0 + k_1a_2$$
$$x_2 = \left(\frac{c}{g_4}L_0^2 + L_0^1k_4a_5 + L_0^0k_3a_4 + k_2a_3\right)x_2^0 - k_1a_1$$
$$x_3 = (\frac{c}{g_5}L_1^2 + \frac{c}{g_4}L_1^1 + k_3a_4)x_3^0 - k_2g_2$$

$$x_4 = (\frac{c}{g_5}L_2^2 + k_4a_5)x_4^0 - k_3g_3$$

$$x_5 = \frac{c}{g_5}x_5^0 - k_4g_4$$

Where $k_i \in Z, 1 \le i \le 4$ arbitrary and

$$g_4l^2 + a_5x_5^0 = g_5$$

$$\Rightarrow a_1(L_0^2 x_1^0) + a_2(L_0^2 x_2^0) + a_3(L_2^2 x_3^0) + a_4(L_2^2 x_4^0) + a_5 x_5^0 = g_5.$$

(6) $\boldsymbol{a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 = c}$

Using the same technique as above we get the general integer solution where

$$x_1 = \left(\frac{c}{g_6}L_0^3 + L_0^2k_5a_6 + L_0^1k_4a_5 + L_0^0k_3a_4 + k_2a_3\right)x_1^0 + k_1a_2$$
$$x_2 = \left(\frac{c}{g_6}L_0^3 + L_0^2k_5a_6 + L_0^1k_4a_5 + L_0^0k_3a_4 + k_2a_3\right)x_2^0 - k_1a_1$$
$$x_3 = (\frac{c}{g_6}L_1^3 + L_1^2k_5a_6 + L_1^1k_4a_5 + k_3a_4)x_3^0 - k_2g_2$$

$$x_4 = (\frac{c}{g_6}L_2^3 + L_2^2k_5a_6 + k_4a_5)x_4^0 - k_3g_3$$

$$x_5 = (\frac{c}{g_6}L_3^3 + k_5a_6)x_5^0 - k_4g_4$$

$$x_6 = \frac{c}{g_6}x_6^0 - k_5a_5$$

Where $k_i \in Z, 1 \leq i \leq 5$ arbitrary and

$$g_5 l^3 + a_6 x_6^0 = g_6$$
$$\Rightarrow a_1(L_0^3 x_1^0) + a_2(L_0^3 x_2^0) + a_3(L_1^3 x_3^0) + a_4(L_2^3 x_4^0) + a_5(L_3^3 x_5^0) + a_6 x_6^0 = g_6$$

Using the pattern for the solutions $\{x_i | i = 1,2,3,..\}$ for $n = 2,3,4,5,6$ let the solution for $n \in N$

i.e. for "$\boldsymbol{a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \cdots . + a_{n-1}x_{n-1} + a_nx_n = c}$" be

$$\boldsymbol{x_1 = \left(\frac{c}{g_n}L_0^{n-3} + L_0^{n-4}k_{n-1}a_n + L_0^{n-5}k_{n-2}a_{n-1} + \cdots + L_0^1k_4a_5 + L_0^0k_3a_4 + k_2a_3\right)x_1^0 + k_1a_2}$$

$$\boldsymbol{x_2 = \left(\frac{c}{g_n}L_0^{n-3} + L_0^{n-4}k_{n-1}a_n + L_0^{n-5}k_{n-2}a_{n-1} + \cdots + L_0^1k_4a_5 + L_0^0k_3a_4 + k_2a_3\right)x_2^0 - k_1a_1}$$

$$\boldsymbol{x_3 = \left(\frac{c}{g_n}L_1^{n-3} + L_1^{n-4}k_{n-1}a_n + L_1^{n-5}k_{n-2}a_{n-1} + \cdots + L_1^2k_5a_6 + L_1^1k_4a_5 + k_3a_4\right)x_3^0 - k_2g_2}$$

$$\boldsymbol{x_4 = \left(\frac{c}{g_n}L_2^{n-3} + L_2^{n-4}k_{n-1}a_n + L_2^{n-5}k_{n-2}a_{n-1} + \cdots + L_2^2k_5a_6 + k_4a_5\right)x_4^0 - k_3g_3}$$

$$\boldsymbol{x_5 = \left(\frac{c}{g_n} L_3^{n-3} + L_3^{n-4} k_{n-1} a_n + L_3^{n-5} k_{n-2} a_{n-1} + \cdots + L_3^3 k_6 a_7 + k_5 a_6\right) x_5^0 - k_4 g_4}$$

… ….

$$\boldsymbol{x_r = \left(\frac{c}{g_n} L_{r-2}^{n-3} + L_{r-2}^{n-4} k_{n-1} a_n + L_{r-2}^{n-5} k_{n-2} a_{n-1} + \cdots + L_{r-2}^{r-2} k_{r+1} a_{r+2} + k_r a_{r+1}\right) x_r^0 - k_{r-1} g_{r-1}}$$

… …

$$\boldsymbol{x_{n-2} = \left(\frac{c}{g_n} L_{n-4}^{n-3} + L_{n-4}^{n-4} k_{n-1} a_n + k_{n-2} a_{n-1}\right) x_{n-2}^0 - k_{n-3} g_{n-3}}$$

$$\boldsymbol{x_{n-1} = \left(\frac{c}{g_n} L_{n-3}^{n-3} + k_{n-1} a_n\right) x_{n-1}^0 - k_{n-2} g_{n-2}}$$

$$\boldsymbol{x_n = \frac{c}{g_n} x_n^0 - k_{n-1} g_{n-1}}$$

Where $\boldsymbol{3 \le r \le n-1}$; $\boldsymbol{k_i \in Z, 1 \le i \le n-1}$;arbitrary and

$$\boldsymbol{a_1 x_1^0 + a_2 x_2^0 = g_2}.$$

$$\boldsymbol{g_i l^{i-2} + a_{i+1} x_{i+1}^0 = g_{i+1} (2 \le i \le n-1)}$$

$$\boldsymbol{g_{n-1} l^{n-3} + a_n x_n^0 = g_n}$$

$$\boldsymbol{\Rightarrow a_1\left(L_0^{n-3} x_1^0\right) + a_2\left(L_0^{n-3} x_2^0\right) + a_3\left(L_1^{n-3} x_3^0\right) + a_4\left(L_2^{n-3} x_4^0\right) + \cdots + a_{n-2}\left(L_{n-4}^{n-3} x_{n-2}^0\right) + a_{n-1}\left(L_{n-3}^{n-3} x_{n-1}^0\right) + a_n x_n^0 = g_n}$$

Now we prove the result by induction i.e for $(n+1)$ varibles. We now find the integer solution of the equation

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \cdots . + a_{n-1}x_{n-1} + a_nx_n + a_{n+1}x_{n+1} = c$$

$$\Rightarrow a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \cdots . + a_{n-1}x_{n-1} + a_nx_n = c - a_{n+1}x_{n+1}$$

As $g_n | c - a_{n+1}x_{n+1}$ so there exist $l \in Z$ such that $c - a_{n+1}x_{n+1} = g_n l \Rightarrow g_n l + a_{n+1}x_{n+1} = c$. Clearly $g.c.d(g_n, a_{n+1}) = g_{n+1} | c$. So $g_n l + a_{n+1}x_{n+1} = c$ has general integer solution

$$l = \frac{c}{g_{n+1}} l^{n-2} + k_n a_{n+1}; \quad x_{n+1} = \frac{c}{g_{n+1}} x_{n+1}^0 - k_n g_n$$

where $g_n l^{n-2} + a_{n+1} x_{n+1}^0 = g_{n+1}$ and $k_n \in Z$ is arbitrary.

Now $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \cdots . + a_{n-1}x_{n-1} + a_nx_n$

$$= (a_1(L_0^{n-3}x_1^0) + a_2(L_0^{n-3}x_2^0) + a_3(L_1^{n-3}x_3^0) + a_4(L_2^{n-3}x_4^0) + \cdots + a_{n-2}(L_{n-4}^{n-3}x_{n-2}^0) + a_{n-1}(L_{n-3}^{n-3}x_{n-1}^0) + a_n x_n^0) l$$

$$\Rightarrow a_1(x_1 - lL_0^{n-3}x_1^0) + a_2(x_2 - lL_0^{n-3}x_2^0) + a_3(x_3 - lL_1^{n-3}x_3^0) + \cdots + a_r(x_1 - lL_{r-2}^{n-3}x_r^0) + \cdots + a_{n-1}(x_{n-1} - lL_{n-3}^{n-3}x_2^0) + a_n(x_n - lx_n^0) = 0$$

$$\Rightarrow a_1X_1 + a_2X_2 + a_3X_3 + \cdots + a_rX_r + \cdots + a_{n-1}X_{n-1} + a_nX_n = 0$$

where $X_1 = x_1 - lL_0^{n-3}x_1^0 = x_1 - \left(\frac{c}{g_{n+1}}l^{n-2} + k_na_{n+1}\right)L_0^{n-3}x_1^0 = x_1 - \left(\frac{c}{g_{n+1}}L_0^{n-2} + L_0^{n-3}k_na_{n+1}\right)x_1^0$; $X_2 = x_2 - \left(\frac{c}{g_{n+1}}L_0^{n-2} + k_na_{n+1}\right)x_2^0$; …; $X_r = x_r - lL_{r-2}^{n-3}x_1^0 = x_r - \left(\frac{c}{g_{n+1}}L_{r-2}^{n-2} + L_{r-2}^{n-3}k_na_{n+1}\right)x_r^0, \dots, X_{n-1} = x_{n-1} - (\frac{c}{g_{n+1}}L_{n-3}^{n-2} + L_{n-3}^{n-3}k_na_{n+1})x_{n-1}^0$

; $X_n = x_n - lx_n^0 = x_n - (\frac{c}{g_{n+1}}L_{n-2}^{n-2} + k_na_{n+1})x_n^0$.

From the result for n varibles as we assume as $g_n|0$, we have

$$X_1 = \left(\frac{0}{g_n}L_0^{n-3} + L_0^{n-4}k_{n-1}a_n + L_0^{n-5}k_{n-2}a_{n-1} + \cdots + L_0^1k_4a_5 + L_0^0k_3a_4 + k_2a_3\right)x_1^0 + k_1a_2.$$

$$\Rightarrow x_1 = \left(\frac{c}{g_{n+1}}L_0^{n-2} + L_0^{n-3}k_na_{n+1} + L_0^{n-4}k_{n-2}a_{n-1} + \cdots + L_0^1k_4a_5 + L_0^0k_3a_4 + k_2a_3\right)x_1^0 + k_1a_2.$$

$$X_2 = \left(\frac{0}{g_n}L_0^{n-3} + L_0^{n-4}k_{n-1}a_n + L_1^{n-5}k_{n-2}a_{n-1} + \cdots + L_0^1k_4a_5 + L_0^0k_3a_4 + k_2a_3\right)x_2^0 - k_1a_1.$$

$$\Rightarrow x_2 = \left(\frac{c}{g_{n+1}}L_0^{n-2} + L_0^{n-3}k_na_{n+1} + L_0^{n-4}k_{n-2}a_{n-1} + \cdots + L_0^1k_4a_5 + L_0^0k_3a_4 + k_2a_3\right)x_2^0 - k_1a_1.$$

$$X_3 = \left(\frac{0}{g_n}L_1^{n-3} + L_1^{n-4}k_{n-1}a_n + L_1^{n-5}k_{n-2}a_{n-1} + \cdots + L_1^2k_5a_6 + L_1^1k_4a_5 + k_3a_4\right)x_3^0 - k_2g_2.$$

$$\Rightarrow x_3 = \left(\frac{c}{g_{n+1}}L_1^{n-2} + L_1^{n-3}k_na_{n+1} + L_1^{n-4}k_{n-1}a_n + \dots + L_1^1k_4a_5 + k_3a_4\right)x_3^0 - k_2g_2.$$

$$X_4 = \left(\frac{0}{g_n}L_2^{n-3} + L_2^{n-4}k_{n-1}a_n + L_2^{n-5}k_{n-2}a_{n-1} + \cdots + L_2^2k_5a_6 + k_4a_5\right)x_4^0 - k_3g_3.$$

$$\Rightarrow x_4 = \left(\frac{c}{g_{n+1}}L_2^{n-2} + L_2^{n-4}k_{n-1}a_n + L_2^{n-5}k_{n-2}a_{n-1} + \cdots + L_2^2k_5a_6 + k_4a_5\right)x_4^0 - k_3g_3.$$

… ….

$$X_r = \left(\frac{0}{g_n}L_{r-2}^{n-3} + L_{r-2}^{n-4}k_{n-1}a_n + L_{r-2}^{n-5}k_{n-2}a_{n-1} + \cdots + L_{r-2}^{r-2}k_{r+1}a_{r+2} + k_ra_{r+1}\right)x_r^0 - k_{r-1}g_{r-1}.$$

$$\Rightarrow x_r = \left(\frac{c}{g_n}L_{r-2}^{n-2} + L_{r-2}^{n-3}k_na_{n+1} + L_{r-2}^{n-4}k_{n-1}a_n + \cdots + L_{r-2}^{r-2}k_{r+1}a_{r+2} + k_ra_{r+1}\right)x_r^0 - k_{r-1}g_{r-1}.$$

… …

$X_{n-1} = \left(\frac{0}{g_n} L_{n-3}^{n-3} + k_{n-1}a_n\right) x_{n-1}^0 - k_{n-1}g_{n-1}.$

$\Rightarrow x_{n-1} = \left(\frac{c}{g_{n+1}} L_{n-3}^{n-2} + L_{n-3}^{n-3}k_n a_{n+1}\right) x_{n-1}^0 - k_{n-1}g_{n-1}.$

$X_n = \frac{0}{g_n} x_n^0 - k_n g_n.$

$\Rightarrow x_n = (\frac{c}{g_{n+1}} L_{n-2}^{n-2} + k_n a_{n+1}) x_n^0 - k_n g_n.$

$x_{n+1} = \frac{c}{g_{n+1}} x_{n+1}^0 - k_n g_n.$

where $3 \leq r \leq n$; $k_i \in Z, 1 \leq i \leq n$; arbitrary and

$a_1 x_1^0 + a_2 x_2^0 = g_2.$

$g_i l^{i-2} + a_{i+1} x_{i+1}^0 = g_{i+1} (2 \leq i \leq n-1).$

$g_{n-1} l^{n-3} + a_n x_n^0 = g_n.$

$g_n l^{n-2} + a_{n+1} x_{n+1}^0 = g_{n+1}.$

$$\Rightarrow \{a_1(L_0^{n-3}x_1^0) + a_2(L_0^{n-3}x_2^0) + a_3(L_1^{n-3}x_3^0) + a_4(L_2^{n-3}x_4^0) + \cdots + a_{n-2}(L_{n-4}^{n-3}x_{n-2}^0) + a_{n-1}(L_{n-3}^{n-3}x_{n-1}^0) + a_n x_n^0\} l^{n-2} + a_{n+1}x_{n+1}^0 = g_{n+1}.$$

$$\Rightarrow a_1(L_0^{n-2}x_1^0) + a_2(L_0^{n-2}x_2^0) + a_3(L_1^{n-2}x_3^0) + a_4(L_2^{n-2}x_4^0) + \cdots + a_{n-2}(L_{n-4}^{n-2}x_{n-2}^0) + a_{n-1}(L_{n-3}^{n-2}x_{n-1}^0) + a_n(L_{n-2}^{n-2}x_{n-1}^0) + a_{n+1}x_{n+1}^0 = g_{n+1}.$$

So the result is true for $n + 1$ also. So by induction the result is true for all natural number.

Example: Find the general integral solutions of the following equation

$\mathbf{2x_1 + 4x_2 + 5x_3 + 10x_4 + 12x_5 + 7x_6 + 9x_7 = 18}$

Solution. Here $a_1 = 2,\ a_2 = 4,\ a_3 = 5,\ a_4 = 10,\ a_5 = 12,\ a_6 = 7,\ a_7 = 9,\ c = 18.$

$a_1x_1 + a_2x_2 = g_2 \Longrightarrow 2x_1 + 4x_2 = 2; g_2 = 2; x_1^0 = -1;\ x_2^0 = 1.$

$g_2 l + a_3 x_3 = g_3 \Longrightarrow 2l + 5x_3 = 1; g_3 = 2; l^0 = -2;\ x_3^0 = 1.$

$g_3 l + a_4 x_4 = g_4 \Longrightarrow l + 10x_4 = 1; g_4 = 1; l^1 = -9;\ x_4^0 = 1.$

$g_4 l + a_5 x_5 = g_5 \Longrightarrow l + 12x_5 = 1; g_5 = 1; l^2 = -11;\ x_5^0 = 1.$

$g_5 l + a_6 x_6 = g_6 \Longrightarrow l + 7x_6 = 1; g_6 = 1; l^3 = -6;\ x_6^0 = 1.$

$g_6 l + a_7 x_7 = g_7 \Longrightarrow l + 9x_7 = 1; g_7 = 1; l^4 = -8;\ x_7^0 = 1.$

Here $n = 7; n - 3 = 4; c = 18$. Therefore using the method of general solutions we have

$$x_1 = \left(\frac{c}{g_7} L_0^4 + L_0^3 k_6 a_7 + L_0^2 k_5 a_6 + L_0^1 k_4 a_5 + L_0^0 k_3 a_4 + k_2 a_3\right) x_1^0 + k_1 a_2.$$

$$x_2 = \left(\frac{c}{g_7} L_0^4 + L_0^3 k_6 a_7 + L_0^2 k_5 a_6 + L_0^1 k_4 a_5 + L_0^0 k_3 a_4 + k_2 a_3\right) x_2^0 - k_1 a_1.$$

$$x_3 = \left(\frac{c}{g_7} L_1^4 + L_1^3 k_6 a_7 + L_1^2 k_5 a_6 + L_1^1 k_4 a_5 + k_3 a_4\right) x_3^0 - k_2 g_2.$$

$$x_4 = \left(\frac{c}{g_7} L_2^4 + L_2^3 k_6 a_7 + L_2^2 k_5 a_6 + k_4 a_5\right) x_4^0 - k_3 g_3 .$$

$$x_5 = \left(\frac{c}{g_7} L_3^4 + L_3^3 k_6 a_7 + k_5 a_6\right) x_5^0 - k_4 g_4.$$

$$x_6 = \left(\frac{c}{g_7} L_4^4 + k_6 a_7\right) x_6^0 - k_5 g_5.$$

$$x_7 = \frac{c}{g_7} x_7^0 - k_6 g_6.$$

After putting the values of a_i, L_j^i, g_i, x_i^0 for i we get

$$\boldsymbol{x_1 = 171072 - 10692k_6 + 2376k_5 - 216k_4 + 20k_3 - 5k_2 + 4k_1.}$$

$$\boldsymbol{x_2 = -171072 + 10692k_6 - 2376k_5 + 216k_4 - 20k_3 + 5k_2 - 2k_1.}$$

$$\boldsymbol{x_3 = 85536 - 5346k_6 + 693k_5 - 108k_4 + 10k_3 - 2k_2.}$$

$$\boldsymbol{x_4 = -9504 + 594k_6 - 77k_5 + 12k_4 - k_3.}$$

$$\boldsymbol{x_5 = 864 - 54k_6 + 7k_5 - k_4.}$$

$$\boldsymbol{x_6 = -144 + 9k_6 - k_5.}$$

$$\boldsymbol{x_7 = 18 - k_6.}$$

Where $\boldsymbol{k_1, k_2, k_3, k_4, k_5, k_6}$ are arbitrary integers.

4. ACKNOWLEDGMENTS

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